Computably Enumerable Sets not Closed under Complement

Suppose $A$ is computably enumerable. Is the complement of $A$, $\overline{A} = \mathbb{N} \setminus A$, necessarily computably enumerable as well? The following theorem and corollary show that the answer is “no.”

**Theorem thy.1.** Let $A$ be any set of natural numbers. Then $A$ is computable if and only if both $A$ and $\overline{A}$ are computably enumerable.

**Proof.** The forwards direction is easy: if $A$ is computable, then $\overline{A}$ is computable as well ($\chi_A = 1 - \chi_{\overline{A}}$), and so both are computably enumerable.

In the other direction, suppose $A$ and $\overline{A}$ are both computably enumerable. Let $A$ be the domain of $\varphi_d$, and let $\overline{A}$ be the domain of $\varphi_e$. Define $h$ by

$$h(x) = \mu s (T(d, x, s) \lor T(e, x, s)).$$

In other words, on input $x$, $h$ searches for either a halting computation of $\varphi_d$ or a halting computation of $\varphi_e$. Now, if $x \in A$, it will succeed in the first case, and if $x \in \overline{A}$, it will succeed in the second case. So, $h$ is a total computable function. But now we have that for every $x$, $x \in A$ if and only if $T(e, x, h(x))$, i.e., if $\varphi_e$ is the one that is defined. Since $T(e, x, h(x))$ is a computable relation, $A$ is computable.

It is easier to understand what is going on in informal computational terms: to decide $A$, on input $x$ search for halting computations of $\varphi_e$ and $\varphi_f$. One of them is bound to halt; if it is $\varphi_e$, then $x$ is in $A$, and otherwise, $x$ is in $\overline{A}$.

**Corollary thy.2.** $K_0$ is not computably enumerable.

**Proof.** We know that $K_0$ is computably enumerable, but not computable. If $K_0$ were computably enumerable, then $K_0$ would be computable by Theorem thy.1.