Suppose \( A \) is computably enumerable. Is the complement of \( A \), \( \overline{A} = \mathbb{N} \setminus A \), necessarily computably enumerable as well? The following theorem and corollary show that the answer is “no.”

**Theorem thy.1.** Let \( A \) be any set of natural numbers. Then \( A \) is computable if and only if both \( A \) and \( \overline{A} \) are computably enumerable.

**Proof.** The forwards direction is easy: if \( A \) is computable, then \( \overline{A} \) is computable as well (\( \chi_A = 1 - \chi_{\overline{A}} \)), and so both are computably enumerable.

In the other direction, suppose \( A \) and \( \overline{A} \) are both computably enumerable. Let \( A \) be the domain of \( \varphi_d \), and let \( \overline{A} \) be the domain of \( \varphi_e \). Define \( h \) by

\[
h(x) = \mu s \left( (T(e, x, s) \vee T(d, x, s)) \right).
\]

In other words, on input \( x \), \( h \) searches for either a halting computation of \( \varphi_d \) or a halting computation of \( \varphi_e \). Now, if \( x \in A \), it will succeed in the first case, and if \( x \in \overline{A} \), it will succeed in the second case. So, \( h \) is a total computable function. But now we have that for every \( x \), \( x \in A \) if and only if \( T(e, x, h(x)) \), i.e., if \( \varphi_e \) is the one that is defined. Since \( T(e, x, h(x)) \) is a computable relation, \( A \) is computable.

It is easier to understand what is going on in informal computational terms: to decide \( A \), on input \( x \) search for halting computations of \( \varphi_e \) and \( \varphi_f \). One of them is bound to halt; if it is \( \varphi_e \), then \( x \) is in \( A \), and otherwise, \( x \) is in \( \overline{A} \).

**Corollary thy.2.** \( K_0 \) is not computably enumerable.

**Proof.** We know that \( K_0 \) is computably enumerable, but not computable. If \( K_0 \) were computably enumerable, then \( K_0 \) would be computable by **Theorem thy.1.**

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**Bibliography**