thy.1 Computably Enumerable Sets are Closed under Union and Intersection

Theorem thy.1. Suppose $A$ and $B$ are computably enumerable. Then so are $A \cap B$ and $A \cup B$.

Proof. ?? allows us to use various characterizations of the computably enumerable sets. By way of illustration, we will provide a few different proofs.

For the first proof, suppose $A$ is enumerated by a computable function $f$, and $B$ is enumerated by a computable function $g$. Let

$$h(x) = \mu y (f(y) = x \lor g(y) = x)$$
$$j(x) = \mu y (f((y)_0) = x \land g((y)_1) = x).$$

Then $A \cup B$ is the domain of $h$, and $A \cap B$ is the domain of $j$.

Here is what is going on, in computational terms: given procedures that enumerate $A$ and $B$, we can semi-decide if an element $x$ is in $A \cup B$ by looking for $x$ in either enumeration; and we can semi-decide if an element $x$ is in $A \cap B$ for looking for $x$ in both enumerations at the same time.

For the second proof, suppose again that $A$ is enumerated by $f$ and $B$ is enumerated by $g$. Let

$$k(x) = \begin{cases} f(x/2) & \text{if } x \text{ is even} \\ g((x-1)/2) & \text{if } x \text{ is odd.} \end{cases}$$

Then $k$ enumerates $A \cup B$; the idea is that $k$ just alternates between the enumerations offered by $f$ and $g$. Enumerating $A \cap B$ is trickier. If $A \cap B$ is empty, it is trivially computably enumerable. Otherwise, let $c$ be any element of $A \cap B$, and define $l$ by

$$l(x) = \begin{cases} f((x)_0) & \text{if } f((x)_0) = g((x)_1) \\ c & \text{otherwise.} \end{cases}$$

In computational terms, $l$ runs through pairs of elements in the enumerations of $f$ and $g$, and outputs every match it finds; otherwise, it just stalls by outputting $c$.

For the last proof, suppose $A$ is the domain of the partial function $m(x)$ and $B$ is the domain of the partial function $n(x)$. Then $A \cap B$ is the domain of the partial function $m(x) + n(x)$.

In computational terms, if $A$ is the set of values for which $m$ halts and $B$ is the set of values for which $n$ halts, $A \cap B$ is the set of values for which both procedures halt.

Expressing $A \cup B$ as a set of halting values is more difficult, because one has to simulate $m$ and $n$ in parallel. Let $d$ be an index for $m$ and let $e$ be an
index for \( n \); in other words, \( m = \varphi_d \) and \( n = \varphi_e \). Then \( A \cup B \) is the domain of the function
\[
p(x) = \mu y \ (T(d, x, y) \lor T(e, x, y)).
\]

In computational terms, on input \( x \), \( p \) searches for either a halting computation for \( m \) or a halting computation for \( n \), and halts if it finds either one. 

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Bibliography