thy.1 Applying the Fixed-Point Theorem

cmp:thy:apf:sec The fixed-point theorem essentially lets us define partial computable functions in terms of their indices. For example, we can find an index e such that for every y,

$$\varphi_e(y) = e + y.$$

As another example, one can use the proof of the fixed-point theorem to design a program in Java or C++ that prints itself out.

Remember that if for each e, we let W_e be the domain of φ_e, then the sequence W_0, W_1, W_2, ... enumerates the computably enumerable sets. Some of these sets are computable. One can ask if there is an algorithm which takes as input a value x, and, if W_x happens to be computable, returns an index for its characteristic function. The answer is "no," there is no such algorithm:

Theorem thy.1. *There is no partial computable function f with the following property: whenever W_e is computable, then $f(e)$ is defined and $\varphi_{f(e)}$ is its characteristic function.*

Proof. Let f be any computable function; we will construct an e such that W_e is computable, but $\varphi_{f(e)}$ is not its characteristic function. Using the fixed point theorem, we can find an index e such that

$$\varphi_e(y) \simeq \begin{cases} 0 & \text{if } y = 0 \text{ and } \varphi_{f(e)}(0) \downarrow= 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That is, e is obtained by applying the fixed-point theorem to the function defined by

$$g(x, y) \simeq \begin{cases} 0 & \text{if } y = 0 \text{ and } \varphi_{f(x)}(0) \downarrow= 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Informally, we can see that g is partial computable, as follows: on input x and y, the algorithm first checks to see if y is equal to 0. If it is, the algorithm computes $f(x)$, and then uses the universal machine to compute $\varphi_{f(x)}(0)$. If this last computation halts and returns 0, the algorithm returns 0; otherwise, the algorithm doesn't halt.

But now notice that if $\varphi_{f(e)}(0)$ is defined and equal to 0, then $\varphi_e(y)$ is defined exactly when y is equal to 0, so $W_e = \{0\}$. If $\varphi_{f(e)}(0)$ is not defined, or is defined but not equal to 0, then $W_e = \emptyset$. Either way, $\varphi_{f(e)}$ is not the characteristic function of W_e, since it gives the wrong answer on input 0. □

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Bibliography