Applying the Fixed-Point Theorem

The fixed-point theorem essentially lets us define partial computable functions in terms of their indices. For example, we can find an index \( e \) such that for every \( y \),

\[
\varphi_e(y) = e + y.
\]

As another example, one can use the proof of the fixed-point theorem to design a program in Java or C++ that prints itself out.

Remember that if for each \( e \), we let \( W_e \) be the domain of \( \varphi_e \), then the sequence \( W_0, W_1, W_2, \ldots \) enumerates the computably enumerable sets. Some of these sets are computable. One can ask if there is an algorithm which takes as input a value \( x \), and, if \( W_x \) happens to be computable, returns an index for its characteristic function. The answer is “no,” there is no such algorithm:

**Theorem thy.1.** There is no partial computable function \( f \) with the following property: whenever \( W_e \) is computable, then \( f(e) \) is defined and \( \varphi_{f(e)} \) is its characteristic function.

**Proof.** Let \( f \) be any computable function; we will construct an \( e \) such that \( W_e \) is computable, but \( \varphi_{f(e)} \) is not its characteristic function. Using the fixed point theorem, we can find an index \( e \) such that

\[
\varphi_e(y) \simeq \begin{cases} 
0 & \text{if } y = 0 \text{ and } \varphi_{f(e)}(0) \downarrow = 0 \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

That is, \( e \) is obtained by applying the fixed-point theorem to the function defined by

\[
g(x, y) \simeq \begin{cases} 
0 & \text{if } y = 0 \text{ and } \varphi_{f(x)}(0) \downarrow = 0 \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

Informally, we can see that \( g \) is partial computable, as follows: on input \( x \) and \( y \), the algorithm first checks to see if \( y \) is equal to 0. If it is, the algorithm computes \( f(x) \), and then uses the universal machine to compute \( \varphi_{f(x)}(0) \). If this last computation halts and returns 0, the algorithm returns 0; otherwise, the algorithm doesn’t halt.

But now notice that if \( \varphi_{f(e)}(0) \) is defined and equal to 0, then \( \varphi_e(y) \) is defined exactly when \( y \) is equal to 0, so \( W_e = \{0\} \). If \( \varphi_{f(e)}(0) \) is not defined, or is defined but not equal to 0, then \( W_e = \emptyset \). Either way, \( \varphi_{f(e)} \) is not the characteristic function of \( W_e \), since it gives the wrong answer on input 0. □

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Bibliography