

## thy.1 Applying the Fixed-Point Theorem

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The fixed-point theorem essentially lets us define partial computable functions in terms of their indices. For example, we can find an index  $e$  such that for every  $y$ ,

$$\varphi_e(y) = e + y.$$

As another example, one can use the proof of the fixed-point theorem to design a program in Java or C++ that prints itself out.

Remember that if for each  $e$ , we let  $W_e$  be the domain of  $\varphi_e$ , then the sequence  $W_0, W_1, W_2, \dots$  enumerates the computably enumerable sets. Some of these sets are computable. One can ask if there is an algorithm which takes as input a value  $x$ , and, if  $W_x$  happens to be computable, returns an index for its characteristic function. The answer is “no,” there is no such algorithm:

**Theorem thy.1.** *There is no partial computable function  $f$  with the following property: whenever  $W_e$  is computable, then  $f(e)$  is defined and  $\varphi_{f(e)}$  is its characteristic function.*

*Proof.* Let  $f$  be any computable function; we will construct an  $e$  such that  $W_e$  is computable, but  $\varphi_{f(e)}$  is not its characteristic function. Using the fixed point theorem, we can find an index  $e$  such that

$$\varphi_e(y) \simeq \begin{cases} 0 & \text{if } y = 0 \text{ and } \varphi_{f(e)}(0) \downarrow = 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That is,  $e$  is obtained by applying the fixed-point theorem to the function defined by

$$g(x, y) \simeq \begin{cases} 0 & \text{if } y = 0 \text{ and } \varphi_{f(x)}(0) \downarrow = 0 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Informally, we can see that  $g$  is partial computable, as follows: on input  $x$  and  $y$ , the algorithm first checks to see if  $y$  is equal to 0. If it is, the algorithm computes  $f(x)$ , and then uses the universal machine to compute  $\varphi_{f(x)}(0)$ . If this last computation halts and returns 0, the algorithm returns 0; otherwise, the algorithm doesn't halt.

But now notice that if  $\varphi_{f(e)}(0)$  is defined and equal to 0, then  $\varphi_e(y)$  is defined exactly when  $y$  is equal to 0, so  $W_e = \{0\}$ . If  $\varphi_{f(e)}(0)$  is not defined, or is defined but not equal to 0, then  $W_e = \emptyset$ . Either way,  $\varphi_{f(e)}$  is not the characteristic function of  $W_e$ , since it gives the wrong answer on input 0.  $\square$

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## Bibliography