tl.1 Semantics for Temporal Logic

aml:tl:sem:sec

Definition tl.1. The basic language of temporal logic contains

1. The propositional constant for falsity $\bot$.
2. The propositional constant for truth $\top$.
3. A denumerable set of propositional variables: p_0, p_1, p_2, ...
4. The propositional connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional), $\leftrightarrow$ (biconditional).
5. Past operators P and H.
6. Future operators F and G.

Later on, we will discuss the potential addition of other kinds of modal operators.

Definition tl.2. *Formulas* of the temporal language are inductively defined as follows:

1. $\bot$ is an atomic formula.
2. $\top$ is an atomic formula.
3. Every propositional variable p_i is an (atomic) formula.
4. If φ is a formula, then $\neg\varphi$ is a formula.
5. If φ and ψ are formulas, then $(\varphi \wedge \psi)$ is a formula.
6. If φ and ψ are formulas, then $(\varphi \vee \psi)$ is a formula.
7. If φ and ψ are formulas, then $(\varphi \rightarrow \psi)$ is a formula.
8. If φ and ψ are formulas, then $(\varphi \leftrightarrow \psi)$ is a formula.
9. If φ is a formula, then $\mathsf{P}\varphi$, $\mathsf{H}\varphi$, $F\varphi$, $\mathsf{G}\varphi$ are all formulas.
10. Nothing else is a formula.

The semantics of temporal logics are given in terms of relational models, as with other kinds of intensional logics.

Definition tl.3. A *model* for temporal language is a triple $\mathfrak{M} = \langle T, \prec, V \rangle$, where

1. T is a nonempty set, interpreted as points in time.
2. $\prec$ is a binary relation on T.

3. V is a function assigning to each propositional variable p a set $V(p)$ of points in time.

When $t \prec t'$ holds, we say that t *precedes* t'. When $t \in V(p)$ we say p is *true at* t.

For now, you will notice that we do not impose any conditions on our precedence relation $\prec$. This means that at present, there are no restrictions on the structure of our temporal models, so we could have models in which time is linear, branching, circular, or has any structure whatsoever.

Just as with normal modal logic, every temporal model determines which formulas count as true at which points in it. We use the same notation "model $\mathfrak{M}$ makes formula φ true at point t" for the basic notion of relational semantics. The relation is defined inductively and is identical to the normal modal case for all non-modal operators.

Definition tl.4. *Truth of a formula* φ *at* t in a $\mathfrak{M}$, in symbols: $\mathfrak{M}, t \Vdash \varphi$, is defined inductively as follows: aml:tl:sem: defn:tmodels

1. $\varphi \equiv \bot$: Never $\mathfrak{M}, t \Vdash \bot$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, t \Vdash \top$.
3. $\mathfrak{M}, t \Vdash p$ iff $t \in V(p)$
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t \nVdash \psi$.
5. $\varphi \equiv (\psi \land \chi)$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t \Vdash \psi$ and $\mathfrak{M}, t \Vdash \chi$.
6. $\varphi \equiv (\psi \lor \chi)$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t \Vdash \psi$ or $\mathfrak{M}, t \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \rightarrow \chi)$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t \nVdash \psi$ or $\mathfrak{M}, t \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, t \Vdash \varphi$ iff either both $\mathfrak{M}, t \Vdash \psi$ and $\mathfrak{M}, t \Vdash \chi$ or neither $\mathfrak{M}, t \Vdash \psi$ nor $\mathfrak{M}, t \Vdash \chi$.
9. $\varphi \equiv \mathsf{P}\psi$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t' \Vdash \psi$ for some $t' \in T$ with $t' \prec t$ aml:tl:sem: defn:sub:mmodels-p
10. $\varphi \equiv \mathsf{H}\psi$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t' \Vdash \psi$ for every $t' \in T$ with $t' \prec t$ aml:tl:sem: defn:sub:mmodels-h
11. $\varphi \equiv \mathsf{F}\psi$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t' \Vdash \psi$ for some $t' \in T$ with $t \prec t'$ aml:tl:sem: defn:sub:mmodels-f
12. $\varphi \equiv \mathsf{G}\psi$: $\mathfrak{M}, t \Vdash \varphi$ iff $\mathfrak{M}, t' \Vdash \psi$ for every $t' \in T$ with $t \prec t'$ aml:tl:sem: defn:sub:mmodels-g

Based on the semantics, you might be able to see that the operators P and H are duals, as well as the operators F and G, such that we could define $\mathsf{H}\varphi$ as $\neg\mathsf{P}\neg\varphi$, and the same with G and F.

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Bibliography