el.1 Truth at a World

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Just as with normal modal logic, every epistemic model determines which formulas count as true at which worlds in it. We use the same notation "model $\mathfrak{M}$ makes formula φ true at world w" for the basic notion of relational semantics. The relation is defined inductively and is identical to the normal modal case for all non-modal operators.

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Definition el.1. *Truth of a formula* φ *at* w in a $\mathfrak{M}$, in symbols: $\mathfrak{M}, w \Vdash \varphi$, is defined inductively as follows:

1. $\varphi \equiv \bot$: Never $\mathfrak{M}, w \Vdash \bot$.
2. $\varphi \equiv \top$: Always $\mathfrak{M}, w \Vdash \top$.
3. $\mathfrak{M}, w \Vdash p$ iff $w \in V(p)$
4. $\varphi \equiv \neg\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$.
5. $\varphi \equiv (\psi \wedge \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$.
6. $\varphi \equiv (\psi \vee \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \Vdash \psi$ or $\mathfrak{M}, w \Vdash \chi$ (or both).
7. $\varphi \equiv (\psi \to \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w \nVdash \psi$ or $\mathfrak{M}, w \Vdash \chi$.
8. $\varphi \equiv (\psi \leftrightarrow \chi)$: $\mathfrak{M}, w \Vdash \varphi$ iff either both $\mathfrak{M}, w \Vdash \psi$ and $\mathfrak{M}, w \Vdash \chi$ or neither $\mathfrak{M}, w \Vdash \psi$ nor $\mathfrak{M}, w \Vdash \chi$.
9. aml:el:trw:defn:sub:mmodels-box $\varphi \equiv \mathsf{K}_a\psi$: $\mathfrak{M}, w \Vdash \varphi$ iff $\mathfrak{M}, w' \Vdash \psi$ for all $w' \in W$ with $R_a w w'$

Here's where we need to think about restrictions on our accessibility relations, though. After all, by clause (9), a formula $\mathsf{K}_a\psi$ is true at w whenever there are no w' with $R_a w w'$. This is the same clause as in normal modal logic; when a world has no successors, all $\Box$-formulas are vacuously true there. This seems extremely counterintuitive if we think about K as representing *knowledge*. After all, we tend to think that there are *no* circumstances under which an agent might know both φ and $\neg\varphi$ at the same time.

One solution is to ensure that our accessibility relation in epistemic logic will always be *reflexive*. This roughly corresponds to the idea that the actual world is consistent with an agent's information. In fact, epistemic logics typically use S5, but others might use weaker systems depending on what exactly they want the K_a relation to represent.

Problem el.1. Consider which of the following hold in Figure 1:

1. $\mathfrak{M}, w_1 \Vdash \neg q$;
2. $\mathfrak{M}, w_1 \Vdash \mathsf{K}_a\neg q$;
3. $\mathfrak{M}, w_1 \Vdash \mathsf{K}_b\neg q$;
4. $\mathfrak{M}, w_2 \Vdash \mathsf{K}_b q \vee \mathsf{K}_b\neg q$;

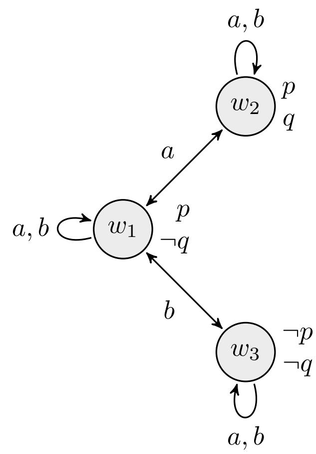

Figure 1: A simple epistemic model.

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5. $\mathfrak{M}, w_2 \Vdash \mathsf{K}_a(\mathsf{K}_b q \lor \mathsf{K}_b \neg q)$;

6. $\mathfrak{M}, w_3 \Vdash \mathsf{E}_{\{a,b\}} \neg q$;

Now that we have given our basic definition of truth at a world, the other semantic concepts from normal modal logic, such as modal validity and entailment, simply carry over, applied to this new way of thinking about the interpretation for the modal operators.

We are now also in a position to give truth conditions for the common knowledge operator C_G. Recall from **??** that the *transitive closure* R^+ of a relation R is defined as

$$R^+ = \bigcup_{n \in \mathbb{N}} R^n,$$

where

$$R^0 = R \text{ and}$$
$$R^{n+1} = \{\langle x, z \rangle : \exists y (R^n xy \land Ryz)\}.$$

Then, where G is a group of agents, we define $R_G = (\bigcup_{b \in G} R_b)^+$ to be the transitive closure of the union of all agents' accessibility relations.

Definition el.2. If $G' \subseteq G$, we let $\mathfrak{M}, w \Vdash \mathsf{C}_{G'} \varphi$ iff for every w' such that $R_{G'} ww'$, $\mathfrak{M}, w' \Vdash \varphi$.

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Bibliography