el.1 Bisimulations

aml:el:bsd:sec One remaining question that we might have about the expressive power of our epistemic language has to do with the relationship between models and the formulas that hold in them. We have seen from our frame correspondence results that when certain formulas are valid in a frame, they will also ensure that those frames satisfy certain properties. But does our modal language, for example, allow us to distinguish between a world at which there is a reflexive arrow, and an infinite chain of worlds, each of which leads to the next? That is, is there any formula A that might hold at only one of these two worlds?

Bisimulation is a relationship that we can define between relational models to say that they have effectively the same structure. And as we will see, it will capture a sense of equivalence between models that can be captured in our epistemic language.

Definition el.1 (Bisimulation). Let $M_1 = \langle W_1, R_1, V_1 \rangle$ and $M_2 = \langle W_2, R_2, V_2 \rangle$ be two relational models. And let $\mathcal{R} \subseteq W_1 \times W_2$ be a binary relation. We say that $\mathcal{R}$ is a *bisimulation* when for every $\langle w_1, w_2 \rangle \in \mathcal{R}$, we have:

1. $w_1 \in V_1(p)$ iff $w_2 \in V_2(p)$ for all propositional variables p.
2. For all agents $a \in A$ and worlds $v_1 \in W_1$, if $R_{1_a} w_1 v_1$ then there is some $v_2 \in W_2$ such that $R_{2_a} w_2 v_2$, and $\langle v_1, v_2 \rangle \in \mathcal{R}$.
3. For all agents $a \in A$ and worlds $v_2 \in W_2$, if $R_{2_a} w_2 v_2$ then there is some $v_1 \in W_1$ such that $R_{1_a} w_1 v_1$, and $\langle v_1, v_2 \rangle \in \mathcal{R}$.

When there is a bisimulation between M_1 and M_2 that links worlds w_1 and w_2, we can also write $\langle M_1, w_1 \rangle \leftrightarrows \langle M_2, w_2 \rangle$, and call $\langle M_1, w_1 \rangle$ and $\langle M_2, w_2 \rangle$ *bisimilar*.

The different clauses in the bisimulation relation ensure different things. Clause 1 ensures that bisimilar worlds will satisfy the same modal-free formulas, since it ensures agreement on all propositional variables. The other two clauses, sometimes referred to as "forth" and "back," respectively, ensure that the accessibility relations will have the same structure.

Theorem el.2. *If $\langle M_1, w_1 \rangle \leftrightarrows \langle M_2, w_2 \rangle$, then for every formula φ, we have that $\mathfrak{M}_1, w_1 \Vdash \varphi$ iff $\mathfrak{M}_2, w_2 \Vdash \varphi$.*

Even though the two models pictured in Figure 1 aren't quite the same as each other, there is a bisimulation linking worlds w_1 and v_1. This bisimulation will also link both w_2 and w_3 to v_2, with the idea being that there is nothing expressible in our modal language that can really distinguish between them. The situation would be different if w_2 and w_3 satisfied different propositional variables, however.

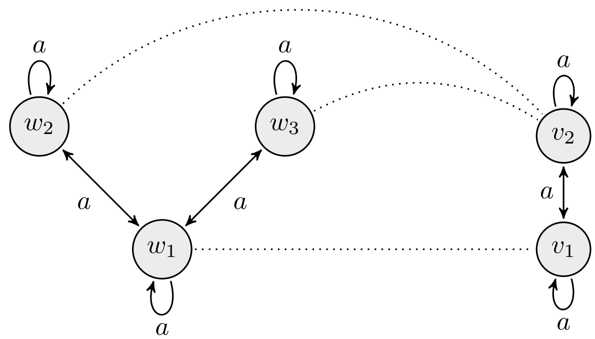

Figure 1: Two bisimilar models.

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Bibliography